

## GEOMETRIC PROPERTIES OF MAPPINGS BETWEEN HYPERSURFACES IN COMPLEX SPACE

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### 0. Introduction

We consider formal, smooth, or real analytic hypersurfaces in  $\mathbb{C}^{n+1}$  and mappings between such hypersurfaces. The mappings we study are either the restrictions of germs of holomorphic mappings, CR mappings, or, more generally, formal holomorphic mappings defined by nonconvergent power series. Let  $M$  be given locally by  $\rho(Z, \bar{Z}) = 0$ , with  $Z \in \mathbb{C}^{n+1}$ ,  $\rho$  real,  $\rho(0) = 0$ ,  $d\rho(0) \neq 0$ . A transversal coordinate for  $M$  is  $Z_{n+1}$  defined by  $\rho(Z, 0) = \alpha(Z)Z_{n+1}$ ,  $\alpha(0) \neq 0$ . If  $H: M \rightarrow M'$  is a holomorphic or formal map between two hypersurfaces  $M, M'$  in  $\mathbb{C}^{n+1}$ , given by  $Z'_i = H_i(Z)$ , and if  $Z'_{n+1}$  is a transversal coordinate for  $M'$ , then  $H_{n+1}$  is called a transversal component of  $H$ .

Our first result in this paper (Theorem 1) shows that if  $M'$  is of finite type (in the sense of Kohn [12] and Bloom-Graham [5]) and  $H$  is of finite multiplicity (as in [3]), then  $(\partial H_{n+1}/\partial Z_{n+1})(0) \neq 0$ , where  $H_{n+1}$  and  $Z_{n+1}$  are transversal. This was proved by Fornaess [9] in the pseudoconvex case using the Hopf Lemma. We also show (Theorem 2) that if  $M$  is essentially finite (as defined in [2], [3] and [6]) and  $H_{n+1}$  does not vanish identically, then  $H$  is of finite multiplicity. Next we show (Theorem 3) that if  $M$  and  $M'$  are essentially finite, then  $H$  is of finite multiplicity if and only if a certain Jacobian determinant associated to  $H$  is nonvanishing.

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Received November 16, 1988. The first author was supported by National Science Foundation Grant DMS-8603176, and the second author DMS-8601260.

In §4 these results are combined and applied to smooth CR mappings (Theorem 4). A relationship between the essential types of  $M$  and  $M'$  and the multiplicity of  $H$  is given in Theorem 5 for smooth CR mappings, similar to that given in [3] for real analytic manifolds.

In §5, we use the previous results, together with the results of [3] to give results on holomorphic extendability of smooth CR mappings between real analytic hypersurfaces with minimal hypotheses (Theorem 6). These results generalize the classical results in the strongly pseudoconvex case due to S. Pinčuk [14] and H. Lewy [13]. More recent work in this direction was given by Baouendi-Jacobowitz-Treves [2] in the diffeomorphic case, the authors in [3] and with S. Bell in [1], and Diederich-Fornaess [8].

§6 deals with global proper holomorphic mappings  $\mathcal{H}$  from one bounded domain  $D$  in  $\mathbb{C}^{n+1}$  to another, both with real analytic boundaries. We assume the mapping extends holomorphically at each point of the boundary of  $D$ . By applying Theorem 4, we show that a transversal derivative of a transversal component of the mapping is necessarily nonvanishing at each point of  $\partial D$  (Theorem 8). We use this result to relate global and local multiplicity of  $\mathcal{H}$  in  $D$  and on the boundary of  $D$ .

Finally, we prove (Theorem 10) that a proper holomorphic self-map of a bounded domain with real analytic boundary is a biholomorphism, generalizing a result of Bedford-Bell [4] in the pseudoconvex case.

Some of these results generalize theorems in  $\mathbb{C}^2$  proved using different methods by the authors jointly with S. Bell in [1].

## 1. Formal hypersurfaces, CR mappings, essential type, and multiplicity

By a *germ of a formal hypersurface*  $M$  at the origin in  $\mathbb{C}^{n+1}$  we shall mean a formal power series of the form

$$(1.1) \quad \rho(Z, \bar{Z}) \sim \sum c_{\alpha\beta} Z^\alpha \bar{Z}^\beta,$$

with the reality condition  $c_{\alpha\beta} = \bar{c}_{\beta\alpha}$ ,  $c_{00} = 0$  and  $\partial\rho(0) \neq 0$ . If  $\rho$  is a real analytic function, then  $\{Z : \rho(Z, \bar{Z}) = 0\}$  is a real analytic hypersurface. If  $M \subset \mathbb{C}^{n+1}$  is a smooth hypersurface defined near 0, then we associate to  $M$  a formal hypersurface by taking  $\rho$  to be the Taylor series of its defining function at the origin. Since a local defining function is determined only up to multiplication by a real nonvanishing function, we will regard two series  $\rho$  and  $\rho_1$  of the form (1.1) as defining the same formal hypersurface if and only if  $\rho_1(Z, \bar{Z}) \sim a(Z, \bar{Z})\rho(Z, \bar{Z})$ , where  $a$  is a real formal power series with  $a(0) \neq 0$ .

After a formal holomorphic change of coordinates we may assume

$$(1.2) \quad \rho(Z, 0) \sim \alpha(Z)Z_{n+1}, \quad \alpha(0) \neq 0.$$

We write  $Z_{n+1} = w$ , and  $(Z_1, \dots, Z_n) = z$ . Then  $w$  is called a *transversal* (formal) coordinate for  $M$ .

**(1.3) Definition.** A formal hypersurface  $M$  at 0 is of *finite type* if  $\rho(z, 0, \bar{z}, 0) \neq 0$ , where  $\rho$  satisfies (1.2).

The reader can easily check that if  $M$  is a smooth embedded hypersurface in  $\mathbb{C}^{n+1}$ , then  $M$  is of finite type at 0 (in the sense of Kohn [12] and Bloom-Graham [5]) if and only if its associated formal hypersurface is of finite type as in Definition (1.3).

**(1.4) Definition.** A formal hypersurface  $M$  at 0 is called *essentially finite* if  $\rho(z, 0, \zeta, 0) \sim \sum a_\alpha(z)\zeta^\alpha$ , with

$$(1.5) \quad \dim_{\mathbb{C}} \mathcal{O}[[z]]/(a_\alpha(z)) < \infty,$$

where  $\rho$  satisfies (1.2), and  $(a_\alpha(z))$  is the ideal generated by the  $a_\alpha(z)$  in the ring of formal power series in  $n$  indeterminates  $\mathcal{O}[[z]]$ . It can be checked (see D'Angelo [6], where a similar definition is given) that definition (1.4) is independent of the choice of formal coordinates. The number given by the left-hand side of (1.5) will be called the *essential type* of  $M$  at 0, written  $\text{ess type}_0 M$ ; it is also independent of the choice of the coordinates.

**(1.6) Definition.** If  $M$  and  $M'$  are two formal hypersurfaces at the origin, a *formal CR map* from  $M$  to  $M'$  is an  $n + 1$  tuple  $H = (H_1, \dots, H_{n+1})$ , where  $H_j(Z) = \sum a_\alpha^j Z^\alpha$  is a formal power series,  $a_\alpha^j \in \mathbb{C}$ ,  $a_0^j = 0$ , such that if  $\rho$  and  $\rho'$  are defining series for  $M$  and  $M'$  respectively, then

$$(1.7) \quad \rho'(H(Z), \bar{H}(\bar{Z})) = b(Z, \bar{Z})\rho(Z, \bar{Z}),$$

where  $b(Z, \bar{Z})$  is a real formal power series.

If we assume the formal coordinates  $Z'$  for  $M'$  satisfy (1.2) with  $Z$  replaced by  $Z'$  and  $\rho$  by  $\rho'$ , then we write  $H_j = F_j$ ,  $1 \leq j \leq n$ , and  $H_{n+1} = G$ , a *transversal component* of  $H$ . If  $\tilde{G}$  is another such transversal component, it is clear that

$$(1.8) \quad G(Z) = b(Z)\tilde{G}(Z), \quad b(0) \neq 0.$$

If the coordinates for  $M$  have been chosen to satisfy (1.2) with the notation  $Z_{n+1} = w$ , then it follows from (1.2) (and a similar formula for  $\rho'$ ) and (1.7) that any transversal component  $G$  satisfies

$$(1.9) \quad G(z, w) = wG_1(z, w),$$

where  $G_1$  is another power series.

**(1.10) Definition.** Suppose that  $H = (F_1, \dots, F_n, G)$  is a formal CR map from  $M$  to  $M'$  where the coordinates for  $M$  and  $M'$  have been chosen as above. Then  $H$  is of *finite multiplicity* if

$$(1.11) \quad \dim_{\mathbb{C}} \mathcal{O}[[z]] / (F_1(z, 0), \dots, F_n(z, 0)) < \infty.$$

One can easily check that this definition, as well as the number defined by (1.11), are independent of the choice of formal coordinates for  $M$  and  $M'$ . The number defined by the left-hand side of (1.11) is called the *multiplicity* of  $H$  at 0, written  $\text{mult}_0 H$ . When  $M$  and  $M'$  are smooth hypersurfaces and  $H$  is a  $C^\infty$  CR map from  $M$  to  $M'$ , it was shown in [3] that there is an associated formal CR map, and the definition of finite multiplicity given there is the same.

When  $M, M'$  and  $H$  are real analytic we have the following results which will be proved in §6.

**(1.12) Proposition.** *If  $M$  is real analytic,  $k$  an integer  $\geq 1$ , and  $U_k$  the germ defined by*

$$(1.13) \quad U_k = \{p \in M : \text{ess type}_p M \geq k\},$$

*then  $U_k$  is a real analytic subvariety of  $M$ .*

**(1.14) Proposition.** *If  $H: M \rightarrow M'$  is a real analytic CR map with  $M$  and  $M'$  real analytic,  $k$  an integer  $\geq 1$ , and  $W_k$  is the germ defined by*

$$(1.15) \quad W_k = \{p \in M : \text{mult}_p H \geq k\},$$

*then  $W_k$  is a real analytic subvariety of  $M$ .*

## 2. Nonvanishing of the differential of a formal CR map

Our main result of this section is the following.

**Theorem 1.** *Let  $H: M \rightarrow M'$  be a formal CR map, where  $M$  and  $M'$  are formal hypersurfaces at the origin in  $\mathbb{C}^{n+1}$ , with  $M'$  of finite type. If  $H$  is of finite multiplicity and  $G$  is a transversal component of  $H$ , then we have*

$$(2.1) \quad \frac{\partial G}{\partial w}(0) \neq 0,$$

*where  $w$  is a transversal coordinate for  $M$ .*

**(2.2) Corollary.** *Let  $\mathcal{H}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$  be a germ of a holomorphic map defined near 0 with  $\mathcal{H}(0) = 0$ . Suppose that  $\mathcal{H}(M) \subset M'$ , where  $M$  and  $M'$  are germs of two real analytic embedded hypersurfaces in  $\mathbb{C}^{n+1}$ , containing 0, with  $M'$  of finite type at 0. If  $\mathcal{H}$  is a finite map (in the sense that the components of  $\mathcal{H}$  have no common zeros near the origin, other than 0), then the gradient of  $\mathcal{H}$  at 0 is nonzero. More precisely, if  $G$  is a transversal*

component of  $H = \mathcal{H}|_M$ , and  $w$  a transversal holomorphic coordinate for  $M$ , then (2.1) holds.

In order to prove Theorem 1, we shall need some notation. As in §1, we write  $H = (F_1, \dots, F_n, G)$ , where  $G$  is a transversal component of  $H$ . We let  $(z, w)$  be formal coordinates for  $M$ , with  $w$  a transversal coordinate. It follows from (1.2) that after a linear change of variables in  $\mathbb{C}^{n+1}$ , we can assume that the defining function  $\rho$  is of the form

$$(2.3) \quad \rho(z, w, \bar{z}, \bar{w}) \sim \frac{w - \bar{w}}{2i} - \varphi \left( z, \bar{z}, \frac{w + \bar{w}}{2} \right),$$

with  $\varphi(0) = 0, \nabla\varphi(0) = 0, \varphi(z, 0, 0) \equiv 0$  and  $\varphi(0, \zeta, 0) \equiv 0$ . By a further formal change of variables as in [2] we may assume that

$$(2.4) \quad \varphi(z, 0, w) \sim \varphi(0, \zeta, w) \sim 0.$$

We shall write  $z = x + iy$  and  $w = s + it, x, y \in \mathbb{R}^n, s, t \in \mathbb{R}$ . Similarly, we may assume that the defining function for  $M'$  is given by

$$(2.5) \quad \rho'(z', \bar{z}', w', \bar{w}') \sim w' - Q(z', \bar{z}', \bar{w}'),$$

where

$$(2.6) \quad Q(z', 0, 0) = Q(0, \zeta', 0) \sim 0, \quad Q(z', 0, \bar{w}') \sim \alpha(z', \bar{w}')\bar{w}',$$

with  $\alpha(0) \neq 0$ .

**(2.7) Lemma.** For  $z, \zeta \in \mathbb{C}^n, s \in \mathbb{C}$  we have

$$(2.8) \quad G(z, s + i\varphi(z, \zeta, s)) \sim Q(F(z, s + i\varphi(z, \zeta, s)), \bar{F}(\zeta, s - i\varphi(z, \zeta, s)), \bar{G}(\zeta, s - i\varphi(z, \zeta, s))).$$

*Proof.* Combining (1.7), (2.3), and (2.5) we obtain

$$(2.9) \quad G(z, w) - Q(F(z, w), \bar{F}(\bar{z}, \bar{w}), \bar{G}(\bar{z}, \bar{w})) \sim b(z, w, \bar{z}, \bar{w}) \left( \frac{w - \bar{w}}{2i} - \varphi \left( z, \bar{z}, \frac{w + \bar{w}}{2} \right) \right).$$

We first replace  $w$  by  $s + i\varphi(z, \bar{z}, s)$  in (2.9), and observe that the right-hand side vanishes. Since we may regard  $z$  and  $\bar{z}$  as independent variables, we may replace  $\bar{z}$  by  $\zeta$  in (2.9), and (2.8) then follows.

We now set a new variable  $\tau$  by

$$(2.10) \quad \tau \sim s - i\varphi(z, \zeta, s).$$

By the implicit function theorem we may find  $R(z, \zeta, \tau)$  satisfying

$$(2.11) \quad s + i\varphi(z, \zeta, s) \sim R(z, \zeta, \tau).$$

(Note that  $R(z, 0, 0) \sim R(0, \zeta, 0) \sim 0$ .) It follows from (2.3) that  $R_\tau(0) \neq 0$ , so that we may uniquely define  $\mu(z, \zeta)$  satisfying

$$(2.12) \quad R(z, s, \mu(z, \zeta)) \sim 0, \quad \mu(z, 0) \sim \mu(0, \zeta) \sim 0.$$

**(2.13) Lemma.** *If  $\frac{\partial G}{\partial w}(0) = 0$ , then for  $k = 1, \dots, n$ ,*

$$(2.14) \quad Q_k(F(z, 0), \bar{F}(\zeta, \mu(z, \zeta)), \bar{G}(\zeta, \mu(z, \zeta)))D(z, \zeta) \sim A(z)\Delta_k(z, \zeta),$$

where

$$Q_k(z', \zeta', w') = \frac{\partial Q}{\partial z'_k}(z', \zeta', w'),$$

$$(2.15) \quad D(z, \zeta) \sim \det[F_{k,z_j}(z, 0) + F_{k,w}(z, 0)R_{z_j}(z, \zeta, \mu(z, \zeta))],$$

$$(2.16) \quad A(z) \sim \sum_l Q^l(F(z, 0), 0, 0)\bar{F}_{l,\bar{w}}(0),$$

where  $Q^l(z', \zeta', w') = (\partial Q / \partial \zeta'_l)(z', \zeta', w')$ , and

$$\Delta_k(z, \zeta) = \det B_k(z, \zeta),$$

$B_k(z, \zeta)$  being obtained by replacing the  $k$ th column in the matrix of the right-hand side of (2.15) by  $R_{z_j}(z, \zeta, \mu(z, \zeta))$ .

*Proof.* We begin by making  $\zeta = 0$  in (2.8) and obtain, using (2.4),

$$(2.17) \quad G(z, s) \sim Q(F(z, s), \bar{F}(0, s), \bar{G}(0, s)).$$

Replacing  $s$  by  $s + i\varphi(z, \zeta, s)$  in (2.17) we get

$$(2.18) \quad G(z, s + i\varphi(z, \zeta, s)) \sim Q(F(z, s + i\varphi(z, \zeta, s)), \bar{F}(0, s + i\varphi(z, \zeta, s)), \bar{G}(0, s + i\varphi(z, \zeta, s))).$$

By subtracting (2.18) from (2.8) we obtain

$$(2.19) \quad Q(F(z, s + i\varphi(z, \zeta, s)), \bar{F}(\zeta, s - i\varphi(z, \zeta, s)), \bar{G}(\zeta, s - i\varphi(z, \zeta, s))) \\ \sim Q(F(z, s + i\varphi(z, \zeta, s)), \bar{F}(0, s + i\varphi(z, \zeta, s)), \bar{G}(0, s + i\varphi(z, \zeta, s))).$$

Making use of (2.10) and (2.11), we obtain from (2.19),

$$(2.20) \quad Q(F(z, R(z, \zeta, \tau)), \bar{F}(\zeta, \tau), \bar{G}(\zeta, \tau)) \\ - Q(F(z, R(z, \zeta, \tau)), \bar{F}(0, R(z, \zeta, \tau)), \bar{G}(0, R(z, \zeta, \tau))) \sim 0.$$

Now we differentiate (2.20) with respect to  $z_j, j = 1, \dots, n$ , set  $\tau = \mu(z, \zeta)$  as defined in (2.12), use (2.6) and the assumption  $\frac{\partial G}{\partial w}(0) = 0$ , to get

$$(2.21) \quad \sum_{k=1}^n Q_k(F(z, 0), \bar{F}(\zeta, \mu(z, \zeta)), \bar{G}(\zeta, \mu(z, \zeta))) \\ \times [F_{k,z_j}(z, 0) + F_{k,w}(z, 0)R_{z_j}(z, \zeta, \mu(z, \zeta))] \\ - A(z)R_{z_j}(z, \zeta, \mu(z, \zeta)) \sim 0.$$

Now the lemma follows by applying Cramer's rule to the system of equations (2.21).

(2.22) **Lemma.** *If  $\frac{\partial G}{\partial w}(0) = 0$  and  $\det(\partial F_j / \partial z_k)(z, 0) \neq 0$ , then*

$$(2.23) \quad Q_k(F(z, \zeta), \overline{F}(\zeta, \mu(z, \zeta)), \overline{G}(\zeta, \mu(z, \zeta))) \sim 0.$$

*Proof.* We shall prove that  $A(Z) \sim 0$ , where  $A(Z)$  is defined by (2.16). The lemma will then follow from Lemma (2.13) and the observation that  $D(z, 0) \sim \det(\partial F_j / \partial z_k)(z, 0) \neq 0$ . We reason by contradiction. Assume

$$(2.24) \quad A(z) \neq 0.$$

We shall show that (2.24) implies that

$$(2.25) \quad \partial_{z^\alpha} D(0, \zeta) \equiv 0 \quad \text{for all } \alpha,$$

contradicting the assumption  $\det(\partial F_j / \partial z_k)(z, 0) \neq 0$ . Let  $l_0 \geq 1$  be minimal so that there exists a multi-index  $\gamma_0$ ,  $|\gamma_0| = l_0$ , such that

$$(2.26) \quad \partial_{z^{\gamma_0}} A(0) \neq 0.$$

By expanding the determinant  $D(z, \zeta)$  given by (2.15) we have

$$(2.27) \quad D(z, \zeta) = D_0(z) + \sum_{k=1}^n F_{k,w}(z, 0) \Delta_k(z, \zeta),$$

where  $D_0(z) = \det(\partial F_j / \partial z_k)(z, 0)$ . After a linear change of the  $F_k$  we may assume (since  $A(z) \neq 0$ ) that

$$(2.28) \quad F_{1,w}(0) = 1 \quad \text{and} \quad F_{j,w}(0) = 0, \quad 2 \leq j \leq n.$$

Therefore we have

$$A(z) \sim Q^1(F(z, 0), 0, 0) \neq 0.$$

Hence also,

$$(2.29) \quad Q_1(0, \overline{F}(\zeta, 0), 0) \neq 0.$$

We return to the proof of (2.25). By (2.14) with  $k = 1$  and (2.29) we conclude that (2.25) holds for  $|\alpha| \leq l_0$ . We claim also that

$$(2.30) \quad \Delta_i(0, \zeta) \sim 0, \quad 1 \leq i \leq n.$$

Indeed by putting  $z = 0$  in (2.27) we get (2.30) for  $i = 1$ . Now apply  $\partial_{z^{\gamma_0}}$  to (2.14) with  $k = 1$ . Since the right-hand side is 0 (by (2.30) with  $i = 1$ ) we obtain  $\partial_{z^{\gamma_0}} D(0, \zeta) \equiv 0$ . Now by applying  $\partial_{z^{\gamma_0}}$  to (2.14) for  $k, k = 2, \dots, n$ , we obtain (2.30) for all  $i$ . Note also that by applying  $\partial_{z^\alpha}$  with  $|\alpha| = l_0$ , we obtain (2.25) for all  $|\alpha| = l_0$ .

Now assume by induction that (2.25) holds for all  $\alpha, |\alpha| \leq l$ , and that

$$(2.31) \quad \Delta_{i,z^\beta}(0, \zeta) \equiv 0 \quad \text{for all } |\beta| \leq l - l_0.$$

We shall prove that (2.25) and (2.31) hold for  $|\alpha| = l + 1$  and  $|\beta| = l - l_0 + 1$ . For such  $\beta$ , apply  $\partial_{z^\beta}$  to (2.27), and put  $z = 0$ . Since  $|\beta| \leq l$ , the inductive assumption and (2.28) give

$$\partial_{z^\beta} D_0(0) + \Delta_{1,z^\beta}(0, \zeta) \sim 0,$$

which shows (2.31) holds with  $i = 1$  and  $|\beta| = l - l_0 + 1$ . Now apply  $\partial_{z^\alpha}$ ,  $|\alpha| = l + 1$ , to (2.14) with  $k = 1$ . Since  $\partial_{z^\beta} A(0) = 0$  if  $|\beta| < l_0$  and  $\partial_{z^\gamma} \Delta_1(0, \zeta) \equiv 0$  if  $|\gamma| \leq l + 1 + l_0$  we obtain

$$Q_1(0, F(\zeta, 0), 0) \partial_{z^\alpha} D(0, \zeta) \sim 0,$$

which proves (2.25) for  $|\alpha| \leq l + 1$ . It remains to prove (2.31) for  $i = 2, \dots, n$  and  $|\beta| = l - l_0 + 1$ . By a rotation we may assume that  $\gamma_0 = (l_0, 0, \dots, 0)$ . We introduce a linear ordering on such multi-indices  $\beta$  by putting

$$(\beta_1, \dots, \beta_n) < (\beta'_1, \dots, \beta'_n)$$

if  $\beta'_1 < \beta_1$  or  $\beta'_1 = \beta_1$  and  $\beta'_2 < \beta_2$ , and so forth. We shall assume by induction that (2.31) holds for all  $\beta$  with  $\beta < \beta^0$  (or we shall make no assumption in case  $\beta^0 = (l - l_0 + 1, 0, \dots, 0)$ ). We shall prove (2.31) for  $\beta^0$ . We apply  $\partial_{z^\gamma}$  to (2.14) for  $k = 2, \dots, n$ , with  $\gamma = (l_0 + \beta_1^0, \beta_2^0, \dots, \beta_n^0)$ , where  $\beta^0 = (\beta_1^0, \dots, \beta_n^0)$ . Since  $\partial_{z^\alpha} D(0, \zeta) \sim 0$  for  $|\alpha| = l + 1$ , the left-hand side vanishes after putting  $z = 0$  and we obtain, by the inductive hypothesis on  $\beta_0$ , that the right-hand side is reduced to

$$\partial_{z_1^{l_0}} A(0) \partial_{z^{\beta^0}} \Delta_k(0, \zeta) \sim 0.$$

This completes the proof of the induction and hence that of Lemma (2.22).

The proof of Theorem 1 will be completed by the following, since the hypothesis that  $H$  is of finite multiplicity implies  $\det(\partial F_j / \partial z_k)(z, 0) \neq 0$  (see [3, Lemma (3.19)]).

**(2.32) Lemma.** *If  $H$  is of finite multiplicity and  $M'$  is of finite type, then for some  $k, 1 \leq k \leq n$ ,*

$$(2.33) \quad Q_k(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), \overline{G}(\zeta, \mu(z, \zeta))) \neq 0.$$

*Proof.* We shall show that if (2.33) does not hold for any  $k$ , then  $Q(z, \zeta, 0) \equiv 0$ , contradicting the assumption that  $M'$  is of finite type. Assume now that for  $k = 1, \dots, n$

$$(2.34) \quad Q_k(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), \overline{G}(\zeta, \mu(z, \zeta))) \equiv 0.$$



In (2.8), we replace  $s - i\varphi(z, \zeta, s)$  by  $\tau$  and  $s + i\varphi(z, \zeta, s)$  by  $R(z, \zeta, \tau)$  as in (2.11); then we replace  $\tau$  by  $\mu(z, \zeta)$ . From (2.12) we obtain (by using (1.9))

$$(2.35) \quad Q(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), \overline{G}(\zeta, \mu(z, \zeta))) \sim 0.$$

Since  $Q_w(0) \neq 0$  by (2.6), we obtain by expanding (2.35)

$$\overline{G}(\zeta, \mu(z, \zeta)) \sim Q(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), 0)\gamma_1(z, \zeta)$$

for some invertible power series  $\gamma_1(z, \zeta)$ . Substituting for  $\overline{G}(\zeta, \mu(z, \zeta))$  in (2.34) and expanding the resulting series yield

$$(2.36) \quad Q_k(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), 0) + Q(F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)), 0)\gamma(z, \zeta) \sim 0,$$

where  $\gamma(z, \zeta)$  is a formal series.

In order to prove that (2.36) implies  $Q(z, \zeta, 0) \sim 0$  we consider first the case where all the power series in (2.36) are convergent. Consider the mapping

$$(2.37) \quad (z, \zeta) \mapsto (F(z, 0), \overline{F}(\zeta, \mu(z, \zeta)))$$

from  $\mathbb{C}^{2n}$  to  $\mathbb{C}^{2n}$ . Since  $\mu(0, \zeta) \equiv 0$  and  $z \mapsto F(z, 0)$  is finite in the sense of [10], we conclude that the same is true for the map defined by (2.37). For  $\varepsilon > 0$  sufficiently small, let  $(z', \zeta') \in \mathbb{C}^{2n}$ ,  $|z'| < \varepsilon$ ,  $|\zeta'| < \varepsilon$ . Let  $N$  be the generic number of inverse images for  $F(z^j, 0) = z'$ ,  $\overline{F}(\zeta^j, \mu(z^j, \zeta^j)) = \zeta'$ . Now replace  $z$  by  $z^j$  and  $\zeta$  by  $\zeta^j$  in (2.36), and sum over  $j$ ,  $1 \leq j \leq N$ , to obtain

$$(2.38) \quad Q_k(z', \zeta', 0) + Q(z', \zeta', 0)\delta(z', \zeta') = 0,$$

where  $\delta(z', \zeta')$  is again holomorphic, since it is a symmetric function of the  $z^j$  and  $\zeta^j$  (see e.g. [10]). Since  $Q(0) = 0$ , from (2.38) by uniqueness for ordinary differential equations we obtain the desired conclusion  $Q(z', \zeta', 0) \equiv 0$ .

For the general case we shall first reduce to the convergent case by truncating the formal series appearing in (2.36), and then show that  $Q(z, \zeta, 0) \equiv 0 \pmod{z_i^p, \zeta_i^p}$  for all integers  $p$ . By an application of Nakayama's lemma (see [3, Lemma (4.3)]) we have, for all  $p$  sufficiently large, the mapping  $(z, \zeta) \rightarrow (F^p(z, 0), \overline{F}^p(\zeta, \mu^p(z, \zeta)))$  is a finite holomorphic mapping of multiplicity  $N$ ,  $N$  independent of  $p$ , where  $F^p$  and  $\mu^p$  are the polynomials obtained from  $F$  and  $\mu$  by dropping all terms containing a factor of the form  $z_i^p$  or  $\zeta_i^p$ . We truncate  $Q$  and  $\gamma$  similarly, and define  $Q_k^p(z', \zeta', 0)$  by

$(Q^p)_{z'_k}(z', \zeta', 0)$ . Then from (2.36) we obtain

$$(2.39) \quad Q_k^p(F^p(z, 0), \bar{F}^p(\zeta, \mu^p(z, \zeta)), 0) + Q^p(F^p(z, 0), \bar{F}^p(\zeta, \mu^p(z, \zeta)), 0)\gamma^p(z, \zeta) \sim 0,$$

$\text{mod } z_i^{p-1}, \zeta_i^p, i = 1, \dots, n$ . Using the same argument as in the holomorphic case we obtain

$$(2.40) \quad Q_k^p(z', \zeta', 0) + Q^p(z', \zeta', 0)\delta_p(z', \zeta') \sim 0,$$

$\text{mod } z_i^k, \zeta_i^k$ , for any  $k \leq (p - 1)/N$ , where we have used the fact that the preimages of  $(z', \zeta')$  under the truncated map are roots of polynomials of degree  $N$  with coefficients holomorphic in  $(z', \zeta')$  (see e.g. [10]). By a similar uniqueness argument as in the holomorphic case we conclude that  $Q(z', \zeta', 0) \equiv 0, \text{ mod } z_i^k, \zeta_i^k, k \leq (p - 1)/N$ . Since  $p$  is arbitrary, we obtain that  $Q(z', \zeta', 0) \sim 0$ , which completes the proof of Lemma (2.32) and hence that of Theorem 1.

### 3. Nonvanishing of the transversal component of a CR map

We show here that a CR map whose transversal component does not vanish identically is of finite multiplicity. More precisely we have

**Theorem 2.** *Let  $H: M \rightarrow M'$  be a formal CR map between two formal hypersurfaces at the origin in  $\mathbb{C}^{n+1}$ . If  $M$  is essentially finite and a transversal component  $G$  of  $H$  does not vanish identically, then  $H$  is of finite multiplicity.*

*Proof.* We shall assume by contradiction that  $H$  is not of finite multiplicity and show that this is impossible, since it implies that  $G \equiv 0$ . As in the proof of Theorem 1, we start with the fundamental identity (2.8) of Lemma (2.7). As before, we make the changes of variables (2.10) and (2.11). Then set  $\tau = 0$  to obtain by using (1.9)

$$(3.1) \quad G(z, \lambda(z, \zeta)) \sim Q(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0),$$

where  $\lambda(z, \zeta) = R(z, \zeta, 0)$ , with  $R$  as defined in (2.11). Again using (1.9) we may expand the left-hand side of (3.1) and write

$$(3.2) \quad \lambda(z, \zeta) \left[ G_w(z, 0) + \sum_{k \geq 2} G_{w^k}(z, 0) \frac{\lambda(z, \zeta)^{k-1}}{k!} \right] \sim Q(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0).$$

We shall need the following two lemmas.

**(3.3) Lemma.** *H is not of finite multiplicity if and only if there is a prime ideal  $\mathcal{I}$  in  $\mathcal{O}[[z]]$  such that*

$$(3.4) \quad (F_1(z, 0), \dots, F_n(z, 0)) \subset \mathcal{I},$$

$$(3.5) \quad \dim_{\mathbb{C}} \mathcal{O}[[z]]/\mathcal{I} = \infty.$$

*Proof.* If  $H$  is of finite multiplicity, then (3.4) and (3.5) are impossible since

$$\dim_{\mathbb{C}} \mathcal{O}[[z]]/(F_1(z, 0), \dots, F_n(z, 0)) < \infty.$$

Conversely, suppose that  $H$  is not of finite multiplicity, and let  $\mathcal{J}_0$  be the radical of  $(F_1(z, 0), \dots, F_n(z, 0))$ . Since  $\mathcal{J}_0 = \text{rad } \mathcal{J}_0$ , we may write

$$(3.6) \quad \mathcal{J}_0 = \bigcap_{j=1}^k \mathcal{P}_j,$$

where each  $\mathcal{P}_j$  is a prime ideal in  $\mathcal{O}[[z]]$ , by the Lasker-Noether decomposition theorem [16, Chapter IV, Theorem 5]. Since  $H$  is not of finite multiplicity,  $\dim_{\mathbb{C}} \mathcal{O}[[z]]/(F_1(z, 0), \dots, F_n(z, 0)) = \infty$ , and hence  $\dim_{\mathbb{C}} \mathcal{O}[[z]]/\mathcal{J}_0 = \infty$  by the Nullstellensatz. We claim that  $\dim_{\mathbb{C}} \mathcal{O}[[z]]/\mathcal{P}_{j_0} = \infty$  for at least one  $j_0$ , otherwise there must be  $N$  for which  $z_i^N \in \mathcal{P}_j$  for all  $j, i = 1, \dots, n$ , contradicting that  $\mathcal{J}_0$  is of infinite codimension in  $\mathcal{O}[[z]]$ . By taking  $\mathcal{I} = \mathcal{P}_{j_0}$ , we prove the lemma.

**(3.7) Lemma.** *Let  $M$  be essentially finite,  $R$  given by (2.11) and  $\lambda(z, \zeta)$  defined by*

$$\lambda(z, \zeta) \sim R(z, \zeta, 0) \sim \sum_{\alpha} a_{\alpha}(\zeta) z^{\alpha}.$$

Then

$$\dim_{\mathbb{C}} \mathcal{O}[[\zeta]]/(a_{\alpha}(\zeta)) < \infty.$$

*Proof.* Let  $\varphi(z, \zeta, 0) \sim \sum_{\alpha} b_{\alpha}(\zeta) z^{\alpha}$  where  $\varphi$  is as in (2.3). By definition,  $M$  is essentially finite if and only if  $\dim_{\mathbb{C}} \mathcal{O}[[\zeta]]/(b_{\alpha}(\zeta)) < \infty$ . We shall show that

$$(3.8) \quad (b_{\alpha}(\zeta)) = (a_{\alpha}(\zeta)).$$

Let  $u(z, \zeta)$  be defined by

$$(3.9) \quad u(z, \zeta) - i\varphi(z, \zeta, u(z, \zeta)) \sim 0, \quad u(z, 0) \sim u(0, \zeta) \sim 0.$$

Then we have

$$(3.10) \quad \lambda(z, \zeta) \sim u(z, \zeta) + i\varphi(z, \zeta, u(z, \zeta)) \sim 2i\varphi(z, \zeta, u(z, \zeta)).$$

From (3.10) it follows that

$$(3.11) \quad \lambda(z, \zeta) \sim 2i\varphi(z, \zeta, 0) + u(z, \zeta)\alpha(z, \zeta),$$

with  $\alpha(0) = 0$ . By (3.9) and (3.11) we obtain

$$(3.12) \quad \lambda(z, \zeta) \sim \varphi(z, \zeta, 0)\beta(z, \zeta),$$

with  $\beta(0) \neq 0$ , which proves (3.8).

**(3.13) Lemma.** *If  $\mathcal{F} \subset \mathcal{O}[[\zeta]]$ ,  $\zeta = (\zeta_1, \dots, \zeta_n)$ , is a prime ideal, and  $\tilde{\mathcal{F}} \subset \mathcal{O}[[\zeta, z]]$ ,  $z = (z_1, \dots, z_p)$ , is the ideal generated by  $\mathcal{F}$  in  $\mathcal{O}[[\zeta, z]]$ , then  $\tilde{\mathcal{F}}$  is again prime.*

*Proof.* By induction we may reduce to the case where  $p = 1$ . In this case any element of  $\mathcal{O}[[\zeta, z]]$  is a power series of the form  $\sum_k c_k(\zeta)z^k$ . If  $(\sum_j d_j(\zeta)z^j)(\sum_k c_k(\zeta)z^k) \in \tilde{\mathcal{F}}$ , then either all the  $d_j$  or all the  $c_j$  are in  $\mathcal{F}$ . Indeed, if not, we may assume  $d_{j_0} \notin \mathcal{F}$  and  $d_j \sim 0$  for  $j < j_0$ , and similarly for  $\sum_k c_k(\zeta)z^k$ , with  $c_{k_0}$ . Then  $d_{j_0}c_{k_0} \in \mathcal{F}$ , contradicting the primality of  $\mathcal{F}$ , and Lemma (3.13) is proved.

We may now complete the proof of Theorem 2. By Lemma (3.3) there is a prime ideal  $\mathcal{F}$  in  $\mathcal{O}[[\zeta]]$  of infinite codimension such that

$$(\bar{F}_1(\zeta, 0), \dots, \bar{F}_n(\zeta, 0)) \subset \mathcal{F}.$$

Let  $\tilde{\mathcal{F}}$  be the ideal generated by  $\mathcal{F}$  in  $\mathcal{O}[[z, \zeta]]$ ,  $z = (z_1, \dots, z_n)$ ,  $\zeta = (\zeta_1, \dots, \zeta_n)$ . Then by Lemma (3.13)  $\tilde{\mathcal{F}}$  is a prime ideal. Since the right-hand side of (3.2) is in  $\tilde{\mathcal{F}}$  (by (2.6)) we conclude that either

$$(3.14) \quad \lambda(z, \zeta) \in \tilde{\mathcal{F}},$$

or

$$(3.15) \quad G_w(z, 0) + \sum_{k \geq 2} G_{w^k}(z, 0) \frac{\lambda(z, \zeta)^{k-1}}{k!} \in \tilde{\mathcal{F}}.$$

By the essential finiteness of  $M$ , together with Lemma (3.7), it follows that (3.14) is impossible. Therefore (3.15) holds, which implies  $G_w(z, 0) \equiv 0$ , since  $\lambda(z, 0) \sim 0$  (by (3.12) and (2.3)). By factoring successive powers of  $\lambda(z, \zeta)$  in (3.2), we prove inductively that  $G_{w^k}(z, 0) \sim 0$  for all  $k$ , reaching the desired contradiction. Hence Theorem 2 is proved.

We include here another consequence of the nonvanishing of the transversal component of a CR map.

**(3.16) Proposition.** *Let  $H: M \rightarrow M'$  be a formal CR map, with  $M, M'$  formal hypersurfaces at the origin in  $\mathbb{C}^{n+1}$ . If  $M$  is of finite type and a transversal component  $G$  of  $H$  does not vanish identically, then  $M'$  is of finite type.*

*Proof.* We begin with identity (3.2). By (3.12), which does not use the assumption that  $M$  is essentially finite,  $M$  is of finite type if and only

if  $\lambda(z, \zeta) \neq 0$ . We claim that the left-hand side of (3.2) does not vanish identically. For if so, we would have

$$(3.17) \quad G_w(z, 0) + \sum_{k \geq 2} G_{w^k}(z, 0) \frac{\lambda(z, \zeta)^{k-1}}{k!} \equiv 0,$$

which implies  $G_w(z, 0) \equiv 0$ , since  $\lambda(z, 0) \equiv 0$ . Factoring successive powers of  $\lambda(z, \zeta)$  would imply  $G_{w^k}(z, 0) \equiv 0$  for all  $k$ , contradicting the assumptions.

**(3.18) Definition.** Let  $H: M \rightarrow M'$  be a formal CR map between two formal hypersurfaces at the origin in  $\mathbb{C}^{n+1}$ . Suppose that  $H = (F_1, \dots, F_n, G)$ , where  $G$  is a transversal component of  $H$ , and  $(z, w)$  are coordinates for  $M$  such that  $w$  is transversal to  $M$ . Then  $H$  is *totally degenerate* if

$$(3.19) \quad \det(\partial F_j / \partial z_k)(z, 0) \sim 0.$$

Using (1.2) and (1.8) one can easily check that condition (3.19) is independent of the choices of coordinates for  $M$  and  $M'$ .

By known results (see e.g. [3] for references) it follows that if  $H$  is of finite multiplicity, then  $H$  is not totally degenerate. However the converse is not true, even if  $M$  or  $M'$  is assumed essentially finite, as shown by the following examples.

**(3.20) Example.** Here  $M$  and  $M'$  are embedded hypersurfaces in  $\mathbb{C}^3$  given by

$$M = \{(z, w) : \text{Im } w - |z_1|^2 - |z_2|^2 = 0\},$$

$$M' = \{(z', w') : \text{Im } w' = 0\}.$$

Let  $H: M \rightarrow M'$  be the holomorphic mapping defined by  $H = (F_1, F_2, G)$ , with  $F_1(z, w) = z_1$ ,  $F_2(z, w) = z_1 z_2$  and  $G = 0$ . Then  $\det(\partial F_j / \partial z_k) = z_2$ , so that  $H$  is not totally degenerate. However,  $\dim \mathcal{O}[[z_1, z_2]] / (z_1, z_1 z_2) = \infty$ , so that  $H$  is not of finite multiplicity.

**(3.21) Example.** Here also  $M$  and  $M'$  are embedded hypersurfaces in  $\mathbb{C}^3$  given by  $M = \{(z, w) : \text{Im } w - |z_1|^2 - |z_1 z_2|^2 = 0\}$  and  $M' = \{(z', w') : \text{Im } w' - |z'_1|^2 - |z'_2|^2 = 0\}$ . Let  $H: M \rightarrow M'$  be the holomorphic mapping defined by  $H = (F_1, F_2, G)$  with  $F_1(z, w) = z_1$ ,  $F_2(z, w) = z_1 z_2$  and  $G = w$ . In this example  $M'$  is essentially finite,  $M$  is of finite type but not essentially finite,  $H$  is not totally degenerate, but not of finite multiplicity.

Under additional assumptions on  $M$  and  $M'$  we can prove that if  $H$  is not totally degenerate, then  $H$  is of finite multiplicity.

**Theorem 3.** *Let  $H: M \rightarrow M'$  be a formal CR map, with  $M$  essentially finite and  $M'$  of finite type. Then  $H$  is of finite multiplicity if and only if  $H$  is not totally degenerate.*

*Proof.* We consider separately the cases where a transversal component  $G$  vanishes identically or not. If  $G \neq 0$ , then by Theorem 2,  $H$  is of finite multiplicity and hence (see above) not totally degenerate.

For the other case assume  $G \equiv 0$ , which implies by Theorem 1 that  $H$  is not of finite multiplicity. Then by (3.1) we have

$$(3.22) \quad Q(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0) \equiv 0$$

with  $\lambda(z, \zeta) = R(z, \zeta, 0)$ . We differentiate (3.22) with respect to  $z_j$ , to obtain the system for  $j = 1, \dots, n$

$$(3.23) \quad \sum_{k=1}^n Q_k(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0) \times [F_{k,z_j}(z, \lambda(z, \zeta)) + F_{k,w}(z, \lambda(z, \zeta))\lambda_{z_j}l(z, \zeta)] \equiv 0.$$

By Cramer's rule,

$$(3.24) \quad Q_k(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0)D(z, \zeta) \equiv 0,$$

where

$$D(z, \zeta) = \det[F_{k,z_j}(z, \lambda(z, \zeta)) + F_{k,w}(z, \lambda(z, \zeta))\lambda_{z_j}(z, \zeta)].$$

If  $D(z, \zeta) \equiv 0$ , then by taking  $\zeta = 0$  and noting again that  $\lambda(z, 0) \equiv 0$  by (3.12), we have  $\det(F_{k,z_j}(z, 0)) \equiv 0$ , which is the desired conclusion. If  $D(z, \zeta) \neq 0$ , then (3.24) implies

$$(3.25) \quad Q_k(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0) \equiv 0.$$

Since  $M'$  is of finite type, we may assume  $Q_{z'^{\alpha_0}\zeta'\beta_0}(0) \neq 0$ . Repeated differentiation of (3.25) with respect to  $z$  (with the assumption  $D(z, \zeta) \neq 0$ ), gives

$$(3.26) \quad Q_{z'^{\alpha_0}}(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0) \equiv 0.$$

Now put  $z = 0$  in (3.26) to obtain

$$(3.27) \quad Q_{z'^{\alpha_0}}(0, \bar{F}(\zeta, 0), 0) \sim 0.$$

Differentiating (3.27) with respect to  $\zeta$ , we find that either  $Q_{z'^{\alpha_0}\zeta'\beta_0}(0) = 0$ , which is impossible, or  $\det(\bar{F}_{k,\zeta_j})(\zeta, 0) \equiv 0$ , completing the proof of the theorem.

**(3.28) Proposition.** *If  $M'$  is of finite type, and  $H$  is not totally degenerate, then  $M$  is of finite type.*

*Proof.* We begin with identity (3.1) from which we obtain, by using (3.12) and (1.9),

$$(3.29) \quad \alpha_1(z, \zeta)\varphi(z, \zeta, 0) \sim Q(F(z, \lambda(z, \zeta)), \bar{F}(\zeta, 0), 0).$$

Now we reason by contradiction. If  $M$  is not of finite type, then  $\varphi(z, \zeta, 0) \equiv 0$  and we obtain (3.22) from (3.29). (In fact we could also replace  $\lambda(z, \zeta)$  by 0.) From this point, the rest of the proof is identical to that of Theorem 3.

We now give an example which shows that we can have  $G \neq 0$ , even  $\frac{\partial G}{\partial w}(0) \neq 0$ , but  $H$  totally degenerate.

**(3.30) Example.** Here  $M$  and  $M'$  are both hypersurfaces in  $\mathbb{C}^3$  given by

$M = \{(z, w) : \text{Im } w - |z_1|^2 = 0\}$ ,  $M' = \{(z, w) : \text{Im } w - |z_1|^2 - |z_2|^2 = 0\}$ , and  $H = (F_1, F_2, G)$  given by  $F_1 = z_1$ ,  $F_2 \equiv 0$ ,  $G = w$ . Then  $M$  is of finite type,  $M'$  is essentially finite,  $H$  is totally degenerate (and hence not of finite multiplicity) but  $\frac{\partial G}{\partial w} \equiv 1$ .

**(3.31) Remark.** When  $M$  and  $M'$  are formal embedded hypersurfaces at 0 in  $\mathbb{C}^2$ , then clearly  $H$  is of finite multiplicity if and only if  $H$  is not totally degenerate, and essential finiteness is equivalent to being of finite type. It has been shown in [1] that if  $M$  and  $M'$  are of finite type, then  $H$  is of finite multiplicity if and only if  $H \neq 0$ .

#### 4. Classification of smooth local CR mappings

In this section we consider a smooth CR mapping  $H: M \rightarrow M'$ , where  $M$  and  $M'$  are embedded smooth hypersurfaces in  $\mathbb{C}^{n+1}$  containing the origin, with  $H(0) = 0$ . We combine Theorems 1, 2, and 3, as well as  $C^\infty$  analogs of some results given in [3] in the real analytic case.

**Theorem 4.** *Let  $H: M \rightarrow M'$  be a smooth CR mapping defined near 0, with  $M$  and  $M'$   $C^\infty$  hypersurfaces in  $\mathbb{C}^{n+1}$ . Let  $w$  be any (formal) transversal coordinate for  $M$  and  $G$  any (formal) transversal coordinate of  $H$ . Assume that  $M$  is essentially finite at 0.*

(i) *If  $G \equiv 0$ , then either  $H$  is not of finite multiplicity at 0 or  $M'$  is not essentially finite.*

(ii) *If  $G \neq 0$ , then  $\frac{\partial G}{\partial w}(0) \neq 0$ ,  $H$  is of finite multiplicity and  $M'$  is essentially finite.*

*In addition if  $M$  and  $M'$  are real analytic and  $H$  is holomorphic, then (ii) holds if and only if  $H$  maps any neighborhood of 0 in  $M$  onto a neighborhood of 0 in  $M'$ .*

*Proof.* Claim (i) is an immediate consequence of Theorem 1. To prove (ii) suppose that  $G \neq 0$ . Then by Proposition (3.16)  $M'$  is of finite type. Also, by Theorem 2,  $H$  is of finite multiplicity. Now we can apply Theorem 1 to prove that  $\frac{\partial G}{\partial w}(0) \neq 0$ . It remains to show that  $M'$  is essentially finite.

To prove this, we will make a slight modification of the proof of Theorem 3 of [3], where this result is proved under the additional assumption that  $M$  and  $M'$  are real analytic.

We begin with identity (3.1) from which we obtain, using  $\frac{\partial G}{\partial w}(0) \neq 0$ ,

$$(4.1) \quad \lambda(z, \zeta)\alpha(z, \zeta) \sim Q(F(z, 0), \bar{F}(\zeta, 0), 0)$$

with  $\alpha(0) \neq 0$ . By means of (3.12) we find

$$(4.2) \quad \alpha_1(z, \zeta)\varphi(z, \zeta, 0) \sim Q(F(z, 0), \bar{F}(\zeta, 0), 0)$$

with  $\alpha_1(0) \neq 0$ . Now (4.2) is very similar to (3.10) of [3]. We proceed as in the proof of Theorem 3 of [3], the main difference being that since  $Q$  is not assumed convergent, we truncate it also. We choose  $k$  as in loc. cit., and, writing  $Q^k, F^k$  and  $\alpha_1^k$  for the truncations of  $Q, F$  and  $\alpha_1$  respectively, we define  $\varphi^{(k)}(z, \zeta)$  by

$$(4.3) \quad \alpha_1^k(z, \zeta)\varphi^{(k)}(z, \zeta) = Q^k(F^k(z), \bar{F}^k(\zeta), 0).$$

Write

$$(4.4) \quad \begin{aligned} Q(z', \zeta', 0) &= \sum b_\alpha(z')\zeta'^\alpha, \\ Q^k(z', \zeta', 0) &= \sum_{|\alpha| \leq k} b_\alpha^k(z')\zeta'^\alpha. \end{aligned}$$

By (4.3), if  $\varphi^{(k)}(z, \zeta) = \sum a_\alpha^{(k)}(z)\zeta^\alpha$ , then

$$(4.5) \quad (a_\alpha^{(k)}(z)) = (b_\alpha^k(F^k(z)))$$

and therefore, in consequence of Lemma (4.5) of [3],

$$(4.6) \quad \dim_{\mathbb{C}} \frac{\mathcal{O}[[z]]}{(b_\alpha^k(z))} = \frac{\dim_{\mathbb{C}} \mathcal{O}[[z]]/(a_\alpha^{(k)}(z))}{\dim_{\mathbb{C}} \mathcal{O}[[z]]/(F_i^k(z))}.$$

We now apply Lemma 4.3 of [3] to conclude that  $\dim_{\mathbb{C}} \mathcal{O}[[z]]/(b_\alpha(z))$  is finite and equal to the left-hand side of (4.6), which is independent of  $k$ . This proves  $M'$  is essentially finite, and hence the proof of Theorem 4 is complete.

**Theorem 5.** *Let  $H: M \rightarrow M'$  be a smooth CR mapping. If either*

- (i)  *$M$  is essentially finite and  $G \neq 0$ , or*
- (ii)  *$M'$  is essentially finite and  $H$  of finite multiplicity,*

*then*

$$(4.7) \quad \text{ess type } M = (\text{mult } H)(\text{ess type } M'),$$

*with all three integers in (4.7) finite.*



*Proof.* If (i) holds, then it follows from Theorem 4 that  $M'$  is essentially finite and  $H$  is of finite multiplicity. If (ii) holds, we claim that  $M$  is essentially finite. If  $M$  and  $M'$  are real analytic, then this claim is Theorem 2 of [3]. In the  $C^\infty$  case we need some modifications. We begin with (4.2) except that now we do not assume  $\alpha_1(0) \neq 0$ . We truncate  $F$  and  $Q$  by choosing  $k$  as follows. Let  $N$  be sufficiently large such that

$$(4.8) \quad (b_\alpha(z)) = (b_\alpha(z) : |\alpha| \leq N).$$

We choose  $p$  so that  $z_i^p \in (b_\alpha(z))$ ,  $i = 1, \dots, n$ , and so that  $z_i^p \in (F_1(z, 0), \dots, F_n(z, 0))$ . Finally choose  $N'$  so that

$$(4.9) \quad (c_\alpha(z)) = (c_\alpha(z) : |\alpha| \leq N'),$$

where  $\alpha_1(z, \zeta)\varphi(z, \zeta, 0) \sim \sum c_\alpha(z)\zeta^\alpha$ . Now choose  $k > \max(N, N', p^2n)$ , and define  $c_\alpha^{(k)}$  by

$$(4.10) \quad \sum c_\alpha^{(k)}(z)\zeta^\alpha = Q^k(F^k(z, 0), \overline{F}^k(\zeta, 0), 0),$$

where  $F^k$  and  $Q^k$  are the truncations of  $F$  and  $Q$ . By the choice of  $p$  and Lemma (4.3) of [3] we have

$$(4.11) \quad z_i^p \in (b_\alpha^k(z)), \quad i = 1, \dots, n,$$

which implies that

$$(4.12) \quad (F_i^k(z, 0))^p \in (b_\alpha^k(F^k(z, 0))).$$

Since  $z_i^p \in (F_j^k(z, 0))$  by Lemma (4.3) of [3], from (4.12) we conclude that for  $i = 1, \dots, n$ ,

$$(4.13) \quad z_i^{p^2n} \in (b_\alpha^k(F^k(z, 0))).$$

Making use of (4.10), Lemma (4.7) in [3], and the fact that  $\overline{F}^k(\zeta, 0)$  is an open map we obtain that

$$(4.14) \quad z_i^{p^2n} \in (c_\alpha^{(k)}(z)) = (c_\alpha(z)),$$

the last equality following from [3, Lemma (4.3)] and (4.9). Since  $(a_\alpha(z)) = (c_\alpha(z))$  where  $\varphi(z, \zeta, 0) \sim \sum a_\alpha(z)\zeta^\alpha$ ,  $M$  is essentially finite.

To complete the proof of Theorem 5, it remains to show (4.7) under the assumption that all three integers are finite. Under this assumption by Theorem 1 we see that  $\frac{\partial G}{\partial w}(0) \neq 0$  so that (4.2) holds with  $\alpha_1(0) \neq 0$ . The desired equality then follows from (4.2) and the end of the argument of the proof of Theorem 4. Hence Theorem 5 is proved.

If  $M$  is a  $C^\infty$  hypersurface in  $\mathbb{C}^{n+1}$  defined by  $\rho(Z, \overline{Z}) = 0$ ,  $\rho(0) = 0$ ,  $d\rho(0) \neq 0$ , it may not be possible to find (convergent) holomorphic

coordinates in  $\mathbb{C}^{n+1}$ ,  $(z_1, \dots, z_n, w)$ , such that  $w$  is a transversal coordinate to  $M$  in the sense of §1. If  $(z_1, \dots, z_n, w)$  are convergent holomorphic coordinates in  $\mathbb{C}^{n+1}$  and  $k \geq 1$  is an integer,  $w$  is called a *transversal holomorphic coordinate for  $M$  of order  $k$*  if

$$(4.15) \quad \rho(z, w, 0, 0) \sim \alpha(z, w)w + \mathcal{O}(|z|^{k+1}, |w|^{k+1}),$$

with  $\alpha$  holomorphic,  $\alpha(0) \neq 0$ . Then  $M$  is parametrized by  $(z, \bar{z}, s)$ , where

$$(4.16) \quad s = \operatorname{Re}(i\alpha(0)w).$$

Given any  $k \geq 1$ , it is easy to show that one can find holomorphic coordinates satisfying (4.15). Moreover if  $M$  is real analytic, one can also take  $k = +\infty$ .

Similarly let  $H: M \rightarrow M'$  be a smooth CR map, and  $M, M'$  be smooth hypersurfaces in  $\mathbb{C}^{n+1}$ . If  $(z'_1, \dots, z'_n, w')$  are holomorphic coordinates for  $M'$  with  $w'$  transversal up to order  $k$ , and  $H = (f_1, \dots, f_n, g)$ , where  $f_i$  and  $g$  are smooth CR functions on  $M$ , then  $g$  is called a *transversal CR component of  $H$  of order  $k$* .

The following is a consequence of Theorem 4.

**(4.17) Corollary.** *Let  $H: M \rightarrow M'$  be a smooth CR map, with  $M$  and  $M'$  smooth hypersurfaces in  $\mathbb{C}^{n+1}$ , and  $M$  essentially finite at the origin. Suppose  $k \geq 1$ , and  $g$  is a transversal CR component of  $H$  of order  $k$ . Then one of the following holds:*

(i)  *$g$  vanishes of order  $k+1$  at 0, and either  $H$  is not of finite multiplicity or  $M'$  is not essentially finite.*

(ii)  *$\frac{\partial g}{\partial s}(0) \neq 0$ , with  $w$  a transversal coordinate for  $M$  of order  $\geq 1$ , and  $s$  given by (4.16). In addition  $H$  is of finite multiplicity at 0, and  $M'$  is essentially finite.*

## 5. Applications to holomorphic extendability of smooth CR mappings

Here we apply §4 to generalize results on holomorphic extendability of smooth CR mappings between real analytic hypersurfaces obtained in [3]. We assume that  $M$  and  $M'$  are real analytic and  $H: M \rightarrow M'$  is a smooth CR mapping. After holomorphic changes of coordinates for  $M$  and  $M'$  we can assume that the coordinates for  $M$  are of the form  $(z, w)$  where  $w$  is a transversal coordinate (of order  $\infty$ ) and  $H = (f_1, \dots, f_n, g)$ ,  $g$  being a transversal CR component of  $H$  (of order  $\infty$ ). Since the Taylor series of  $g$  coincides with  $G(z, w)$  restricted to  $M$  (see §1 and [3, §2] for further details), it follows that  $G \equiv 0$  if and only if  $g$  is flat at 0.

**Theorem 6.** *Let  $H: M \rightarrow M'$  be a smooth CR map,  $H(0) = 0$ , where  $M$  and  $M'$  are real analytic hypersurfaces in  $\mathbb{C}^{n+1}$ , and  $g$  is a transversal CR component of order  $\infty$ . Then  $H$  extends holomorphically to a neighborhood of  $0$  in  $\mathbb{C}^{n+1}$  if any one of the following conditions holds.*

- (i)  *$M$  is essentially finite, and  $g$  is not flat at  $0$ .*
- (ii)  *$M'$  is essentially finite, and  $H$  is of finite multiplicity at  $0$ .*
- (iii)  *$M'$  is essentially finite, and  $H$  is not totally degenerate at  $0$ .*

*Proof.* Suppose first that (i) holds. Since  $g$  is not flat at  $0$ ,  $G \not\equiv 0$ . Then from Theorem 4 it follows that condition (ii) also holds. On the other hand, since  $H$  is of finite multiplicity, it is not totally degenerate, and hence condition (ii) implies (iii).

It remains to prove that (iii) implies holomorphic extendability. An inspection of the proof of Theorem 1 of [3] shows that extendability holds if  $M$  is of finite type,  $M'$  is essentially finite, and  $H$  is not totally degenerate and also satisfies the following condition: If  $H = (F_1, \dots, F_n, G)$  and  $P(z'_1, \dots, z'_n)$  is a holomorphic function (with constant coefficients), then

$$(5.1) \quad p(F_1(z, 0), \dots, F_n(z, 0)) \equiv 0 \text{ implies } p \equiv 0.$$

As is well known, if (5.1) does not hold, then  $\det(\partial F_j / \partial z_k) \equiv 0$ , which would contradict the assumption that  $H$  is not totally degenerate. Finally, by Proposition (3.28), since  $M'$  is of finite type and  $H$  not totally degenerate, it follows that  $M$  is of finite type. Hence, all the conditions for extendability of [3] are satisfied if (iii) holds.

**(5.2) Remark.** By Theorems 4 and 5, conditions (i) and (ii) of Theorem 6 are, in fact, equivalent. Theorem 1 of [3] already contains the result that extendability holds under condition (ii) of Theorem 6. However, condition (iii) is weaker than (i) and (ii), as is shown by Example (3.21), where  $H$  is not totally degenerate, but is not of finite multiplicity, with  $M'$  essentially finite. It should be noted that in that example the map  $H: M \rightarrow M'$  is not finite-to-one.

### 6. Multiplicities of proper holomorphic mappings

In this section we consider a proper holomorphic mapping  $\mathcal{H}: D \rightarrow D'$ , where  $D$  and  $D'$  are bounded domains in  $\mathbb{C}^{n+1}$  with real analytic boundaries such that  $\mathcal{H}$  extends holomorphically in a neighborhood of  $\bar{D}$ . By Theorem 6, this is satisfied, if  $\mathcal{H}$  is assumed to be smooth up to the boundary, and no transversal component of  $\mathcal{H}$  is flat at any point of  $M = \partial D$ .

**Theorem 7.** *Let  $\mathcal{H} : D \rightarrow D'$  be as above. Then at every point  $q \in \partial D$  we have*

$$(6.1) \quad \frac{\partial G}{\partial w}(q) \neq 0,$$

where  $G$  and  $w$  are respectively a transversal component of  $\mathcal{H}$  and a transversal coordinate for  $M$  at  $q$ . In addition,  $\mathcal{H}|_M$  is of finite multiplicity at every point  $q \in \partial D$ , and  $\mathcal{H}$  extends to a proper holomorphic map from an open neighborhood  $D_1$  of  $\bar{D}$  to  $D'_1$ , an open neighborhood of  $\bar{D}'$ .

*Proof.* It suffices to apply Theorem 4 with  $M = \partial D$  near  $q$ ,  $M' = \partial D'$  and  $H = \mathcal{H}|_M$ . By the theorem of Diederich-Fornaess [7],  $M$  does not contain any complex analytic variety and hence is essentially finite. We may eliminate the case  $G \equiv 0$  since this would contradict the assumption that  $\mathcal{H}$  is proper and hence finite. By Theorem 4 we conclude (6.1) as well as the finite multiplicity of  $\mathcal{H}|_M$ . Using (1.9) we also see that  $\mathcal{H}$  is a finite map at every point of  $\partial D$ . To prove that  $\mathcal{H}$  extends to a proper map we cover  $\partial D$  with finitely many balls in which  $\mathcal{H}$  is a proper map. It remains to prove that  $\mathcal{H}$  extends as a proper holomorphic map from an open neighborhood of  $D_1$  of  $\bar{D}$  to  $D'_1$ , an open neighborhood of  $\bar{D}'$ . Let  $\rho'(Z', \bar{Z}')$  be a real valued defining function for  $D'$ , i.e.,  $D' = \{Z' \in \mathbb{C}^{n+1}, \rho'(Z', \bar{Z}') < 0\}$ ,  $d\rho' \neq 0$ . Using (6.1) it suffices to choose  $\varepsilon > 0$  sufficiently small and to take

$$D_1 = \{Z \in \mathbb{C}^{n+1}, \rho'(\mathcal{H}(Z), \overline{\mathcal{H}(Z)}) < \varepsilon\}.$$

For  $p \in D_1$ , a neighborhood of  $\bar{D}$ , we define  $m(\mathcal{H}, p)$ , the multiplicity of  $\mathcal{H}$  at  $p$ , by

$$m(\mathcal{H}, p) = \dim_{\mathbb{C}} \mathcal{O}[[Z - p]] / (\mathcal{H}_j(z) - \mathcal{H}_j(p)).$$

For  $k \geq 1$  we denote by  $V_k$  the set of points in  $D_1$  of multiplicity  $\geq k$ , i.e.,

$$(6.2) \quad V_k = \{p \in D_1 : m(\mathcal{H}, p) \geq k\}.$$

Clearly  $V_1 = D_1$  and  $V_2 = \{Z \in D_1 : \det(\partial \mathcal{H}_j / \partial z_k)(Z) = 0\}$ . More generally we have the following.

**(6.3) Proposition.** *For every  $k \geq 2$ ,  $V_k$  is an analytic set, more precisely, there exist  $f_1^k, \dots, f_{N_k}^k$ , holomorphic in  $D_1$ , such that*

$$(6.4) \quad V_k = \{p \in D_1 : f_j^k(p) = 0, 1 \leq j \leq N_k\}.$$

This result is, no doubt, known in the folklore. However, lacking an explicit reference we include a proof here.

We begin with the following lemmas, which will also be needed in the proofs of Propositions (1.12) and (1.14).

**(6.5) Lemma.** *Let  $f_1, \dots, f_J$  be formal power series in  $\mathcal{O}[[z_1, z_2, \dots, z_n]]$  such that  $(f) = (f_1, \dots, f_J)$  is of finite codimension. Then for any integer  $k \geq 1$  the following are equivalent:*

- (i)  $\text{codim}(f) \leq k$ ,
- (ii)  $\text{codim}(f_1, \dots, f_J, z_1^{k+1}, \dots, z_n^{k+1}) \leq k$ .

*Proof.* Assume (i). Then for  $j = 1, \dots, n$ , we have  $1, z_j, \dots, z_j^k$  are linearly dependent in  $\mathcal{O}[[z]]/(f)$ . Therefore  $z_j^k \in (f)$ . Hence  $(f_1, \dots, f_J) = (f_1, \dots, f_J, z_1^{k+1}, \dots, z_n^{k+1})$ , which proves (ii). Conversely, assume (ii). Again we conclude  $z_j^k \in (f_1, \dots, f_J, z_1^{k+1}, \dots, z_n^{k+1})$ . Hence by a version of Nakayama's Lemma (see e.g. Lemma (4.3) of [3]) we obtain

$$(f) = (f_1, \dots, f_J, z_1^{k+1}, \dots, z_n^{k+1}),$$

which implies (i), and hence the proof of Lemma (6.5) is complete.

**(6.6) Lemma.** *Let  $k \geq 1$  be an integer. There exist  $N_1$ , a positive integer, and  $K_1, \dots, K_{N_1}$  polynomials in  $J(k+1)^n$  variables such that if*

$$f_j(z) = \sum_{\alpha} a_{\alpha}^j z^{\alpha}, \quad 1 \leq j \leq J,$$

then

$$(6.7) \quad \text{codim}(f) > k$$

if and only if

$$(6.8) \quad K_{\mu}(a_{\alpha}^j) = 0, \quad 1 \leq j \leq J, \quad 1 \leq \mu \leq N_1, \quad \alpha_i \leq k, \quad 1 \leq i \leq n.$$

*Proof.* Consider the space  $\mathcal{P}_k[z_1, \dots, z_n]$  of polynomials of the form  $p(z) = \sum a_{\alpha} z^{\alpha}$ ,  $a_{\alpha} = 0$ , if  $\alpha_i > k$  for some  $i$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Let  $N$  be the dimension of  $\mathcal{P}_k$  as a vector space over  $\mathbb{C}$ , and  $e_1, \dots, e_N$  a basis of  $\mathcal{P}_k$  consisting of monomials. We write

$$(6.9) \quad f_j^{(k)}(z) = \sum a_{\alpha}^j z^{\alpha}, \quad \text{where } a_{\alpha}^j = \begin{cases} \alpha_{\alpha}^j & \text{if } \alpha_i \leq k \text{ for all } i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Assume  $\text{codim}(f) = p \leq k$ . It follows from Lemma (6.5) that there exist  $\{g_1, \dots, g_p\} \subset \{e_1, \dots, e_N\}$  such that every  $r(z) \in \mathcal{P}_k$  can be decomposed,

$$(6.10) \quad r(z) = \sum_{j=1}^p a_j g_j(z) + \sum_{j=1}^J f_j^{(k)}(z) h_j(z) \quad \text{mod } (z_i^{k+1}),$$

with  $a_j \in \mathbb{C}$  and  $h_j \in \mathcal{P}_k$ . By decomposing (6.10) in terms of the basis  $\{e_1, \dots, e_N\}$  we find  $N$  equations in  $JN + p$  unknowns,  $a_j$  and the coefficients of the  $h_j$ . Since the coefficients of  $r$  are arbitrary, we must have an  $N \times N$  nonzero determinant involving the coefficients of the  $f_j^{(k)}$ .

The lemma will follow by considering the finitely many possible subsets  $\{g_1, \dots, g_p\} \subset \{e_1, \dots, e_N\}$ ,  $p \leq k$ .

Now to complete the proof of Proposition (6.3), it suffices to use Lemma (6.6) with  $J = n$  and to replace  $a_\alpha^j$  by  $\frac{1}{\alpha!}(\partial^\alpha \mathcal{H}_j / \partial z^\alpha)(p)$  in (6.8).

Since the proofs of Propositions (1.12) and (1.14) also make use of Lemma (6.6) we shall give them here.

*Proof of Proposition (1.12).* Let  $\rho(Z, \bar{Z})$  be a real analytic defining function for  $M$ . We assume  $(\partial \rho / \partial Z_{n+1})(0) \neq 0$ . By the implicit function theorem there exists a holomorphic function  $\theta(\zeta', p)$  in  $2n + 1$  variables defined near the origin satisfying

$$(6.11) \quad \rho(p, \zeta', \theta(\zeta', p)) \equiv 0$$

with  $\theta(\bar{p}', p) = \bar{p}_{n+1}$  for  $p \in M$ . We have used the notation  $\zeta = (\zeta', \zeta_{n+1})$ ,  $p = (p', p_{n+1})$ .

If we write

$$\rho(Z, \zeta', \theta(\zeta', p)) = \sum_{\alpha} a_{\alpha}(Z, p, \bar{p})(\zeta' - \bar{p}')^{\alpha},$$

then it is easy to see that, for  $p \in M$ ,

$$(6.12) \quad \text{ess type}_p M = \dim_{\mathbb{C}} \mathcal{O}[[Z - p]] / (a_{\alpha}(Z, p, \bar{p})).$$

Note that the functions  $a_{\alpha}(Z, p, q)$  are holomorphic, and by (6.11), satisfy  $a_{\alpha}(p, p, \bar{p}) = 0$ .

By the Noetherian Theorem there exists an integer  $J_0$  such that

$$(a_{\alpha}(Z, p, q)) = (a_{\alpha}(Z, p, q), |\alpha| \leq J_0)$$

as ideals in  $\mathcal{O}[[z - p, p, q]]$ . Hence there is a number  $J$  and multi-indices  $\alpha^{(1)}, \dots, \alpha^{(J)}$  such that

$$(a_{\alpha}(Z, p, \bar{p})) = (a_{\alpha^{(j)}}(Z, p, \bar{p})), \quad 1 \leq j \leq J,$$

as ideals in  $\mathcal{O}[[Z - p]]$  for all  $p$  near the origin. We may now apply Lemma (6.6) as in the proof of Proposition (6.3) to show that the points of type  $\geq k$  satisfy real analytic equations.

*Proof of Proposition (1.14).* If  $M$  is defined by  $\rho(Z, \bar{Z}) = 0$  near the origin, with  $\rho$  real analytic, and if  $H(Z) = (H_1(Z), \dots, H_{n+1}(Z))$ , it is easy to see that for  $p \in M$ , near the origin,

$\text{mult}_p H$

$$= \dim_{\mathbb{C}} \mathcal{O}[[Z - p]] / (H_1(Z) - H_1(p), \dots, H_{n+1}(Z) - H_{n+1}(p), \rho(Z, \bar{p})).$$

We again use Lemma (6.6), as in the proofs of Propositions (6.3) and (1.12) to show that the points of type  $\geq k$  satisfy real analytic equations.

We shall also need the following, which shows that the analytic set  $V_2$  must cross the boundary of  $D$ . More precisely we have

**(6.13) Proposition.** *Let  $\mathcal{H}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ ,  $M$  and  $M'$  be real analytic hypersurfaces containing the origin such that  $M$  is essentially finite and  $\mathcal{H}(M) \subset M'$ . Let  $\Delta(z) = \det(\partial \mathcal{H}_j / \partial z_k)(Z)$ , and  $V$  be the germ of the zero set of  $\Delta$ . If  $\Delta(0) = 0$ , then for every neighborhood  $\Omega$  of 0 in  $\mathbb{C}^{n+1}$  we have*

$$(6.14) \quad V \cap \Omega^+ \neq \emptyset \quad \text{and} \quad V \cap \Omega^- \neq \emptyset,$$

where  $\Omega^\pm = \{Z \in \Omega: \rho(Z, \bar{Z}) \gtrless 0\}$ ,  $\rho$  a defining function for  $M$ .

*Proof.* By Theorem 4, either a transversal component  $G$  of  $\mathcal{H}$  vanishes identically, in which case (6.14) is obvious, or else  $\frac{\partial G}{\partial w}(0) \neq 0$  with  $w$  a transversal holomorphic coordinate for  $M$ , and  $\mathcal{H}|_M$  is of finite multiplicity. We assume the latter conclusions and write  $\mathcal{H} = (F_1, \dots, F_n, G)$ . Since  $\mathcal{H}|_M$  of finite multiplicity implies  $\det((\partial F_j / \partial z_k)(z, 0)) \neq 0$  (see [3, Lemma (3.19)]) and by (1.9)

$$\Delta(z, 0) = \det \left( \frac{\partial F_j}{\partial z_k}(z, 0) \right) \cdot \left( \frac{\partial G}{\partial w}(z, 0) \right),$$

we have, after a linear change of variables in  $z_1, \dots, z_n$ ,

$$\frac{\partial^k}{\partial z_1^k} \Delta(0, 0) \neq 0.$$

Hence, by the Weierstrass Preparation Theorem, the set  $V$  is given by an equation of the form

$$(6.15) \quad z_1^k + \sum_{j=0}^{k-1} a_j(z', w) z_1^j = 0,$$

where  $z' = (z_2, \dots, z_n)$ , and the  $a_j$  are holomorphic and vanish at 0. Let  $z_1^{(1)}, \dots, z_1^{(k)}$  be the roots of (6.15) at  $(z', w)$ . Define the function  $\tilde{\rho}(z', w, \bar{z}', \bar{w})$  by

$$(6.16) \quad \tilde{\rho}(z', w, \bar{z}', \bar{w}) = \sum_{j=1}^k \rho(z_1^{(j)}, z', w, \bar{z}_1^{(j)}, \bar{z}', \bar{w}).$$

Since the right-hand side of (6.16) is a symmetric function of the roots of (6.15), it is clear that  $\tilde{\rho}$  is a real-valued real analytic function. We claim that

$$(6.17) \quad \frac{\partial}{\partial w} \tilde{\rho}(0) \neq 0.$$

If so, this will prove that  $\tilde{\rho}$  changes sign in any neighborhood of 0 in  $\mathbb{C}^n$  and hence so does  $\rho$  on  $V$ . This will prove (6.12).

To prove the claim (6.17) it suffices to compute  $\tilde{\rho}(0, w, 0, 0)$ , i.e., by taking  $z' = 0$  and replacing  $\bar{z}$  and  $\bar{w}$  by 0. By (1.2) we have

$$\rho(z, w, 0, 0) = a(w)w + \sum_{\substack{\alpha \geq 1 \\ \beta \geq 1}} a_{\alpha\beta} z_1^\alpha w^\beta + O(|z'|),$$

with  $a(w)$  holomorphic and  $a(0) \neq 0$ . Therefore using (6.16) we obtain

$$\tilde{\rho}(0, w, 0, 0) = ka(w)w + \sum_{\beta \geq 1} h_\beta(w)w^\beta,$$

with

$$h_\beta(w) = \sum_{j=1}^k \sum_{\alpha \geq 1} a_{\alpha\beta} (z_1^{(j)}(0, w))^\alpha,$$

where  $z_1^{(j)}$  are the roots of (6.15) as above. It is clear that  $h_\beta$  is holomorphic since it is a symmetric function of the roots, and we also have  $h_\beta(0) = 0$ ,  $\beta \geq 1$ . Hence (6.17) is proved.

**Theorem 8.** *Let  $\mathcal{H} : D \rightarrow D'$  be as above, and  $H : \partial D \rightarrow \partial D'$  be the restriction of the extension of  $\mathcal{H}$  to  $\partial D$ . Let  $m(H, q)$  be the multiplicity of  $H$ , as a CR mapping, at  $q$ . Then there exists  $\varepsilon > 0$  such that*

$$(6.18) \quad \sup_{p \in D_\varepsilon} m(\mathcal{H}, p) \leq \sup_{q \in \partial D} m(H, q),$$

where  $D_\varepsilon = \{z \in D : d(z, \partial D) < \varepsilon\}$ .

*Proof.* Since  $m(\mathcal{H}, p)$  is the topological multiplicity of  $\mathcal{H}$  in a sufficiently small neighborhood around  $p$  (see e.g. [10]), for each  $p \in \bar{D}$  there exists a neighborhood  $U$  of  $p$  such that  $m(\mathcal{H}, q) \leq m(\mathcal{H}, p)$  for all  $q \in U$ . By compactness of  $\bar{D}$  we conclude that the set  $V_k$  defined by (6.2) is empty for  $k$  sufficiently large, say  $k \geq k_0$ . Let  $E$  be the set of all isolated points in  $\bigcup_{2 \leq k \leq k_0} V_k \cap D$ . From local properties of analytic sets and the compactness of  $\bar{D}$  it follows that  $E$  is finite. We choose  $\varepsilon$  sufficiently small so that  $D_\varepsilon \cap E = \emptyset$ . Hence any component in  $D$  of  $\bigcup_{2 \leq k \leq k_0} V_k \setminus E$  is not compact. Therefore its closure must intersect  $\partial D$ . This shows

$$(6.19) \quad \sup_{p \in D_\varepsilon} m(\mathcal{H}, p) \leq \sup_{q \in \partial D} m(\mathcal{H}, q).$$

We need to compare  $m(\mathcal{H}, q)$  and  $m(H, q)$  for  $q \in \partial D$ . By Theorem 7 it follows that a transversal component  $G$  of  $\mathcal{H}$  satisfies  $\frac{\partial G}{\partial w}(q) \neq 0$ , where  $w$  is a transversal holomorphic coordinate for  $M$  and therefore (using (1.9)),  $m(\mathcal{H}, q) = m(H, q)$  for all  $q \in \partial D$ . This proves Theorem 8.

The following example shows that (6.18) does not hold if  $D_\varepsilon$  is replaced by  $D$ .



**(6.20) Example.** Let  $D$  and  $D'$  be contained in  $\mathbb{C}^2$  and be defined by

$$D = \{(z, w) \in \mathbb{C}^2 : |z^3 + zw|^2 + |w|^2 < 1\},$$

$$D' = \{(z', w') \in \mathbb{C}^2 : |z'|^2 + |w'|^2 < 1\}.$$

Let  $\mathcal{H} = (z^3 + zw, w)$ . Clearly  $\mathcal{H}$  is a proper map from  $D$  into  $D'$ . Then

$$m(\mathcal{H}, (z, w)) = \begin{cases} 1 & \text{if } 3z^2 + w \neq 0, \\ 2 & \text{if } 3z^2 + w = 0, z \neq 0, \\ 3 & \text{if } z = w = 0. \end{cases}$$

Therefore

$$(6.21) \quad \sup_{p \in D} m(\mathcal{H}, p) = 3 > \sup_{p \in \partial D} m(H, p) = 2.$$

The next example shows that inequality (6.18) of Theorem 8 can be strict for every positive  $\varepsilon$ .

**(6.22) Example.** Let  $D$  and  $D'$  be domains in  $\mathbb{C}^3$  given by

$$D = \{(z_1, z_2, w) \in \mathbb{C}^3 : |z_1^3 + wz_1^2|^2 + |z_2|^2 + |w - 1|^2 < 1\},$$

$$D' = \{(z'_1, z'_2, w') \in \mathbb{C}^3 : |z'_1|^2 + |z'_2|^2 + |w' - 1|^2 < 1\},$$

and let  $\mathcal{H} = (z_1^3 + wz_1^2, z_2, w)$ . Then

$$m(\mathcal{H}, (z, w)) = \begin{cases} 1 & \text{if } z_1 \neq 0 \text{ and } 3z_1 + 2w \neq 0, \\ 2 & \text{if } (z_1 = 0 \text{ and } w \neq 0) \text{ or } (z_1 \neq 0 \text{ and } w = -\frac{3}{2}z_1), \\ 3 & \text{if } z_1 = w = 0. \end{cases}$$

Therefore  $V_3 \cap \bar{D} = (0)$  and  $V_3 \cap D = \emptyset$ . Hence

$$\sup_{q \in \partial D} m(H, q) = 3 > \sup_{p \in D} m(\mathcal{H}, p) = 2.$$

Note that the set  $V_3 = \{z_1 = w = 0\}$  stays entirely in  $\mathbb{C}^3 \setminus D$  and intersects  $\bar{D}$  only at the origin. This is in contrast with the result for  $V_2$  given by Proposition (6.13).

**Theorem 9.** *If  $m(H, q) = 1$  for all  $q \in \partial D$ , then  $m(\mathcal{H}, p) = 1$  for all  $p \in D$ . Also, if  $\sup_{q \in \partial D} m(H, q) = 2$ , then (6.18) becomes an equality.*

*Proof.* In order to prove the first statement, note that the hypothesis is equivalent to  $V_2 \cap \partial D = \emptyset$ , since  $V_2$  is the zero of the Jacobian determinant. The compactness of  $\bar{D}$  implies that  $V_2 \cap D = \emptyset$  also. For the second statement we use Proposition (6.13) to conclude that if  $V_2 \cap \partial D \neq \emptyset$ , then  $V_2 \cap D \neq \emptyset$  (and also  $V_2 \cap (\mathbb{C}^{n+1} \setminus D) \neq \emptyset$ ). Therefore we obtain the opposite inequality of (6.18), and hence the theorem is proved.

We shall now deal with questions of global multiplicity. Suppose  $\mathcal{H} : D \rightarrow D'$  is as above. For  $p \in D$ , we let  $\mu(\mathcal{H}, p)$  be the number of preimages

in  $D$  of  $\mathcal{H}(p)$ . Similarly, if  $q \in \partial D$  we denote by  $\mu(H, q)$  the number of preimages of  $H(q)$  in  $\partial D$ . We have the following corollary of Theorem 7.

**(6.23) Corollary.** *If  $\mathcal{H}: D \rightarrow D'$  is as above, we have*

$$(6.24) \quad \sup_{q \in \partial D} \mu(H, q) = \sup_{p \in D} \mu(\mathcal{H}, p).$$

*In particular, if  $\mathcal{H}$  is 1-1 on  $D$ , then  $\mathcal{H}$  is 1-1 on a neighborhood of  $\bar{D}$ .*

*Proof.* By Theorem 8,  $\mathcal{H}$  extends to a proper mapping from  $D_1$  to  $D'_1$ . The set  $E$  of points  $p$  such that

$$\sup_{q \in D} \mu(\mathcal{H}, q) = \sup_{q \in D_1} \mu(\mathcal{H}, q) = \mu(\mathcal{H}, p)$$

is a dense open set in  $D_1$  for which  $E \cap \partial D$  is also dense in  $\partial D$  (see e.g. [16]). This proves (6.24). In particular, if  $\mathcal{H}$  is 1-1 on  $D$ , then  $\mathcal{H}$  is also 1-1 on  $D_1$ , proving the corollary.

Our last result deals with proper self-maps.

**Theorem 10.** *Let  $D$  be an open domain in  $\mathbb{C}^{n+1}$  with real analytic boundary, and let  $\mathcal{H}$  be a proper holomorphic self-map of  $D$ . If  $\mathcal{H} \in C^\infty(\bar{D})$ , and a transversal component  $G$  of  $\mathcal{H}$  is not flat, at every point  $q \in \partial D$  then  $\mathcal{H}$  extends as a biholomorphism from an open neighborhood of  $\bar{D}$  into another.*

*Proof.* By Theorem 6(i), we may assume that  $\mathcal{H}$  extends holomorphically in a neighborhood of  $\bar{D}$ . The proof then proceeds as in the proof of Theorem 4 of [1] with the following modification. The set  $s(d)$  of points of  $\partial D$  of type  $\geq d$  must be replaced by the set of points in  $\partial D$  of essential type  $\geq d$ , which is again a real subvariety of Proposition (1.12). Also, the identity relating the multiplicity of  $H$  at  $p$  and the types of  $\partial D$  at  $p$  and  $H(p)$  must be replaced by the corresponding identity on essential types given by (4.7) of Theorem 5. The rest of the proof of Theorem 10 is the same as the proof of Theorem 6 in [1].

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